Copyright Notice

The equivalence of bounded nondeterminacy and continuity.

Unbounded nondeterminacy is presented by the function "any natural number" such that

$$wp(\text{"x:= any natural number"}, 0 \leq x) = T$$
$$wp(\text{"x:= any natural number"}, x \leq k) = F \quad \text{for } \underline{\text{all}} \ k .$$

Program S is continuous --see Chapter 9 of "A Discipline of Programming", where this property is called Property 5-- means that for any infinite sequence of predicates C_0 , C_1 , C_2 , ... such that

$$\text{for } r \geq 0 \qquad C_r \Rightarrow C_{r+1} \qquad \text{for all states}$$

we have for all states

$$wp(S, (\underline{E}\ r: r \geq 0: C_r)) = (\underline{E}\ s: s \geq 0: wp(S, C_s)) \qquad (1)$$

and in the same chapter I have shown that all programs that could be written in my programming language fragment --with finite (!) guarded command sets-- are continuous.

It is further shown that the program "x:= any natural number" is not continuous --and, hence, cannot be written in that programming language fragment-- . For the sake of completeness, we repeat the proof. Assume the program S: "x:= any natural number" to be continuous. We then have:

$$\begin{aligned} T &= wp(S, 0 \leq x) \\ &= wp(S, (\underline{E}\ r: r \geq 0: 0 \leq x \leq r)) \\ &= (\underline{E}\ s: s \geq 0: wp(S, 0 \leq x \leq s)) \\ &= (\underline{E}\ s: s \geq 0: F) = F \end{aligned}$$

a contradiction that leads to the conclusion that "x:= any natural number" cannot be continuous, i.e. that continuity implies bounded nondeterminacy.

In the sequel of this note we shall show that the inverse holds as well, viz. that the existence of a noncontinuous program implies the inclusion of unbounded nondeterminacy. (The following argument was suggested to me by C.S.Scholten almost instantaneously when I had posed the problem.)

Assume the existence of a program S and an infinite sequence of predicates C_r such that $C_r \Rightarrow C_{r+1}$, such that (1) does not hold. Because in (1) the right-hand side implies the left-hand side trivially, this means that we assume

$$wp(S, (\underline{E}\ r: r \geq 0: C_r))\ \underline{and}\ \underline{non}\ (\underline{E}\ s: s \geq 0: wp(S, C_s)) =$$
$$wp(S, (\underline{E}\ r: r \geq 0: C_r))\ \underline{and}\ (\underline{A}\ s: s \geq 0: \underline{non}\ wp(S, C_s)) \qquad (2)$$

to be different from F .

Consider now the program

$$S;\ x:= (\underline{MIN}: k: C_k)$$

started in an initial state satisfying (2). Because the initial state satisfies $wp(S, (\underline{E}\ r: r \geq 0: C_r))$, this program terminates and is guaranteed to establish $0 \leq x$. On the other hand, the assumption that for some K it is certain to establish $x \leq K$ means that S is certain to establish C_K , a conclusion that is incompatible with the second term of (2). Hence its nondeterminacy is unbounded. (The fact that our program of unbounded nondeterminacy is not a total program, but only defined for initial states satisfying (2) is here not relevant: the essential thing is that (2) differs from F , i.e. that the set of states satisfying (2) is not empty.)

Here we have established the <u>equivalence</u> of continuity and the boundedness of nondeterminacy. In EWD673 we have established the equivalence between the boundedness of nondeterminacy and the equality between weak and strong termination. Hence the three criteria

1) continuity or not
2) nondeterminacy bounded or not
3) weak and strong termination equivalent or not

are three different aspects of the <u>same</u> dichotomy. All this is very satisfying. (The arguments are so simple that, presumable, this is already known. But it was new for me, and I like the arguments.)

Plataanstraat 5
5671 AL NUENEN
The Netherlands

prof.dr.Edsger W.Dijkstra
BURROUGHS Research Fellow