

D. A. Turner's reply.

In reply to EWD759, D. A. Turner sent me the following proof by returning mail.

"Thank you for your "somewhat open letter", which arrived yesterday. You pose several tasks - the order in which I have decided to tackle them is to establish first the precise formal relationship between f and g .

First some notation. I shall call the elements of a list f , f_0, f_1, f_2 etc. and I shall use the notation

$$[h(i)]_{i=A}^B$$

for the list $[h(A), h(A+1), \dots, h(B)]$.

Now I define a function "upto"

$$\text{upto } i \text{ } f = \text{least } j \geq 0 \text{ such that } f_j > i \quad (\text{upto}.0)$$

The claim to be established is that

$$g = [\text{upto } i \text{ } f]_{i=0}^{\infty} \quad (\text{Theorem 1})$$

From upto.0 we deduce the following two propositions, which could be considered a SASL' definition of upto

$$a > i \vdash \text{upto } i (a:f) = 0 \quad (\text{upto.1})$$

$$a \leq i \vdash \text{upto } i (a:f) = 1 + \text{upto } i f \quad (\text{upto.2})$$

We have also your definition of the function "k"

$$p \leq y \vdash k \ x \ y \ (p:q) = k \ (x+1) \ y \ q \quad (k.1)$$

$$p > y \vdash k \ x \ y \ (p:q) = x : k \ x \ (y+1) \ (p:q) \quad (k.2)$$

From these four propositions we shall deduce the following generalization of Theorem 1.

Theorem 0. $k \ x \ y \ f = [x + \text{upto } i \ f]_{i=y}^{\infty}$

Proof by structural induction on f , which is an infinite list of integers.

case Ω_L (Note - we need to distinguish between Ω_L , the undefined element in the space to which f belongs, and Ω_I , the undefined integer. The relationship between them is $\Omega_L = [\Omega_I]_0^{\infty}$)

$k \ x \ y \ \Omega_L = \Omega_L$ from k.1, k.2 by case exhaustion
whereas

$$[x + \text{upto } i \ \Omega_L]_{i=y}^{\infty}$$

$$= [x + \Omega_I]_{i=y}^{\infty} \quad \text{from upto.1, upto.2}$$

$$= [\Omega_I]_{i=y}^{\infty} \quad \text{properties of } \Omega$$

$$= \Omega_L \quad \text{as required}$$

$$\begin{aligned}
& \text{case } p: f \\
& k \times y (p: f) \\
& = [x]^{p-y} ++ k \times p (p: f) \quad \text{by repeated appl. of k.2} \\
& = [x]^{p-y} ++ k (x+1) p f \quad \text{by k.1} \\
& = [x]^{p-y} ++ [(x+1) + \text{upto } i f]_{i=p}^{\infty} \quad \text{ex hyp.} \\
& = [x]^{p-y} ++ [x + (1 + \text{upto } i f)]_{i=p}^{\infty} \quad \text{properties of } + \\
& = [x]^{p-y} ++ [x + \text{upto } i (p: f)]_{i=p}^{\infty} \quad \text{by upto.2} \\
& = [x + \text{upto } i (p: f)]_{i=y}^{\infty} \quad \text{by upto.1 and} \\
& \quad \quad \quad \text{rearranging}
\end{aligned}$$

QED Theorem 0.

Whence, since $g = k \ 0 \ 0 \ f$, we have immediately

$$\text{Theorem 1} \quad g = [\text{upto } i f]_{i=0}^{\infty}$$

Also you asked me to establish that g is A) ascending and B) unbounded, given appropriate assumptions about f . This now follows easily from the above. (Relaxing the level of formality somewhat) we have:

A) From upto.0 it follows (by transitivity of " $>$ ") that "upto $i f$ " is an ascending function of i . Therefore, whatever the nature of f , g is ascending.

B) Let us define " f is unbounded" to mean: "for any $N \geq 0$, there is a $j \geq 0$ such that $f_j > N$ ".

Assume f is unbounded (f ascending not relevant).
Then, from upto. 0,

upto i f is defined for all $i \geq 0$

Given any $N \geq 0$, define $j = \max\{f_0, \dots, f_N\}$
then upto j $f = g_j$ exists, and by construction $j > N$
So g too is unbounded.

* * *

So far Turner's reply. I like Turner's proof, and in view of the fact that Turner answered me by returning mail it would be misplaced to complain too much about the fact that, in "case $p:f$ " the definitely less interesting case $p < y$ hasn't been dealt with explicitly.

I am slightly uneasy in "case Ω_L " - particularly after the parenthetical remark explaining the difference between Ω_L and Ω_I - about the justification "from k.1, k.2 by case exhaustion". My uneasiness is certainly caused by lack of familiarity how to deal with Ω . Take

$$\text{funny } (p:q) = \begin{array}{l} \text{if } p \geq 10 \rightarrow 1: \text{funny } q \\ \text{if } p < 10 \rightarrow 1: \text{funny } q \end{array}$$

Is $\text{funny } \Omega_L = \Omega_L$? Or is $\text{funny } \Omega_L = \text{ones}$
(with $\text{ones} = 1: \text{ones}$)? I expect the first answer,

though I would prefer the second one, if I am giving full weight to the remark (in [1], pg 37)

"The first point to be made is that in reasoning about SASL programs, Ω can be treated just like any other value as regards being substitutable in equations."
 Perhaps I have failed to fathom the complete depth of the constraint "as regards being substitutable in equations".

The correspondence was triggered by remarks in [1] such as the recommendation of applicative programming (pg. 14):

"The proofs (like the programs themselves) are very much shorter than the proofs of the corresponding imperative programs."

I had my doubts, which have not been dispelled by the comparison of Turner's proof with the one given in EWD758.

[1] Turner, D.A. "Program Proving and Applicative Languages", August 1980

Plataanstraat 5
 5671 AL NUENEN
 The Netherlands

27 December 1980
 prof.dr. Edsger W. Dijkstra
 Burroughs Research Fellow