Why the importance of continuity seems to be overrated

Without an appeal to continuity and without using fixed-point induction, we shall prove the following theorem.

Theorem Let $(C, <)$ be a well-founded set. Let predicates B and P, statement S, and function t be such that

$$[P \Rrightarrow t \text{ \underline{in} } C] \qquad (0)$$

and with fresh "thought variable" y

$$[B \wedge P \Rrightarrow wp(\text{"}y := t\text{"}, wp(S, P \wedge t < y))] \quad . \qquad (1)$$

Then $$[P \Rrightarrow wp(\text{"}\underline{\text{do}}\ B \rightarrow S\ \underline{\text{od}}\text{"}, true)] \quad , \qquad (2)$$

in which the right-hand side is defined as the strongest solution of

$$X: [(B \wedge wp(S, X)) \vee \neg B \equiv X] \quad . \qquad (3)$$

Proof. Equation (3) has a strongest solution since $wp(S, ?)$ is conjunctive and, hence, monotonic. Let X be the strongest solution of (3). Since we can conclude from (0)

$$[P \Rrightarrow (\underline{E} x : x \text{ \underline{in} } C : t \le x)] \quad ,$$

(2) – i.e. $[P \Rrightarrow X]$ – is proved by demonstrating

$$[P \wedge (\underline{E} x : x \text{ \underline{in} } C : t \le x) \Rrightarrow X]$$

or, equivalently,

$$(\underline{A}\, x : x\ \underline{in}\ C : [P \wedge t \leq x \Rrightarrow X]) \qquad . \qquad (4)$$

In view of C's well-foundedness, (4) will be shown by mathematical induction, i.e. for an x in C, we shall derive $[P \wedge t \leq x \Rrightarrow X]$ under the hypothesis $[P \wedge t < x \Rrightarrow X]$.

We observe for any Z

$[Z \equiv B \wedge P \wedge t \leq x]$

$\Rightarrow$ {(1)}

$[Z \Rrightarrow wp(\text{"}y:=t\text{"}, wp(S, P \wedge t < y)) \wedge t \leq x]$

= {Axiom of Assignment; conjunctivity of wp}

$[Z \Rrightarrow wp(\text{"}y:=t\text{"}, wp(S, P \wedge t < y) \wedge y \leq x)]$

= {$[wp(S,Q) \wedge y \leq x \equiv wp(S, Q \wedge y \leq x)]$ for any Q since y and x are thought variables}

$[Z \Rrightarrow wp(\text{"}y:=t\text{"}, wp(S, P \wedge t < y \wedge y \leq x))]$

$\Rightarrow$ {monotonicity of wp; transitivity of <}

$[Z \Rrightarrow wp(\text{"}y:=t\text{"}, wp(S, P \wedge t < x))]$

= {y is a thought variable}

$[Z \Rrightarrow wp(S, P \wedge t < x)]$

$\Rightarrow$ {Hypothesis and monotonicity of wp}

$[Z \Rrightarrow wp(S, X)]$.

Eliminating Z, we conclude under the hypothesis

$$[B \wedge P \wedge t \leq x \Rrightarrow B \wedge wp(S, X)] \qquad ;$$

furthermore we have

$$[\neg B \wedge P \wedge t \leq x \Rrightarrow \neg B] \qquad .$$

Hence

$$[P \wedge t \leq x \Rrightarrow (B \wedge wp(S,X)) \vee \neg B]$$

and, since X is a solution of (3),

$$[P \wedge t \leq x \Rrightarrow X] \quad .$$

(End of Proof.)

The theorem is well-known for or-continuous wp(S,?) and natural t. The continuity permits us to write the strongest solution of (3) in closed form, viz. as the limit of a weakening chain. I (=EWD) used this expression a decade ago to prove the restricted theorem, but that proof was by no means simpler than our current one.

The above proof casts serious doubts on the supposed need of fancy things such as transfinite induction for reasoning about programs with unbounded nondeterminacy (as we might, for instance, encounter in an abstract program containing the unrefined statement "establish P" or with fair interleaving of the atomic actions of concurrent programs). This is a very nice thought.

drs. A.J.M. van Gasteren
BP Venture Research Fellow
Dept. of Mathematics and
Computing Science
University of Technology
5600 MB EINDHOVEN
The Netherlands

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prof. dr. Edsger W. Dijkstra
Burroughs Research Fellow
Plataanstraat 5
5671 AL NUENEN
The Netherlands