On the theorem of Pythagoras

For the theorem of Pythagoras, I start from Coxeter's formulation ("Introduction to Geometry", p. 8):

"In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the two catheti."

Let us play a little bit with that formulation. In a triangle with sides a, b, and c — different from 0 so as to make its angles well-defined — we introduce the usual nomenclature α, β, and γ for their respective opposite angles. (We introduced one angle name so as to be able to express right-angledness, and the other two for reasons of symmetry.)

A formal expression of Coxeter's formulation is

$$\gamma = \pi/2 \quad \Rightarrow \quad a^2 + b^2 = c^2 \quad .$$

Besides the nomenclature we introduced, this formulation contains the (transcendental!) constant π. Fortunately we can eliminate it thanks to

$$\pi = \alpha + \beta + \gamma \quad .$$

Elementary arithmetic yields the equivalent formulation

$$\alpha + \beta = \gamma \quad \Rightarrow \quad a^2 + b^2 = c^2 \quad .$$

Isn't that nicely symmetric? It immediately suggests — at least to me — the strengthening

$$(0) \qquad \alpha + \beta = \gamma \quad \equiv \quad a^2 + b^2 = c^2 \quad .$$

(This will turn out to be a theorem.) We get an equivalent formulation by negating both sides:

$$\alpha + \beta \neq \gamma \quad \equiv \quad a^2 + b^2 \neq c^2 \quad .$$

But $x \neq y \equiv x < y \vee x > y$, and the latter disjuncts are mutually exclusive. Remembering that the larger angle is opposite to the larger side, is it bold to guess

(1) $\alpha + \beta < \gamma \quad \equiv a^2 + b^2 < c^2$ and

(2) $\alpha + \beta > \gamma \quad \equiv a^2 + b^2 > c^2$?

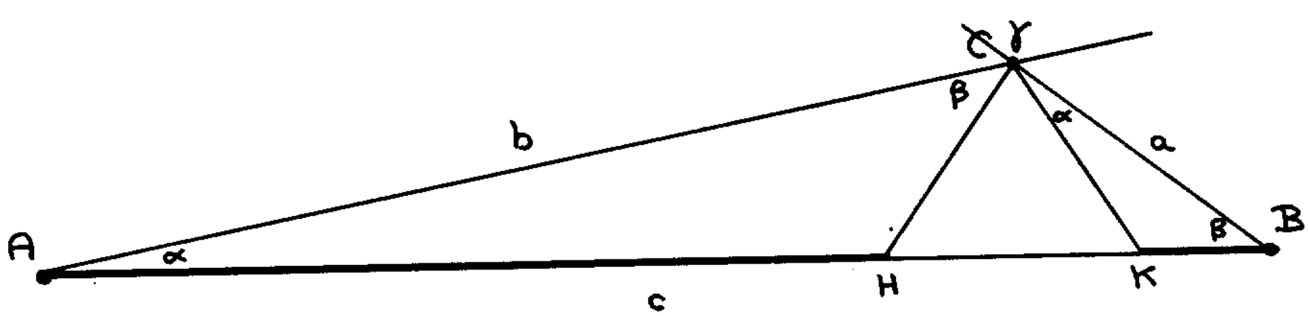
Bold perhaps, but not unreasonable.

Note that (0), (1) and (2) are not independent: from any two of them the third can be derived. They can be jointly formulated in terms of the function sgn - read "signum" - given by

$$sgn.0 = 0 \quad \wedge \quad (sgn.x = 1 \equiv x > 0) \quad \wedge \quad (sgn.x = -1 \equiv x < 0) ,$$

viz. $\quad sgn.(\alpha + \beta - \gamma) = sgn.(a^2 + b^2 - c^2) \quad .$

Consider now the following figure. We have drawn



the case $\alpha + \beta < \gamma$, in which the triangles ΔCKB and ΔAHC, of disjoint areas, don't cover the whole of ΔACB; denoting the area of ΔXYZ by "XYZ" we have in this case

$$CKB + AHC < ACB \quad .$$

In the case $\alpha+\beta=\gamma$, H and K coincide and we have

$$CKB + AHC = ACB \quad ,$$

and in the case $\alpha+\beta>\gamma$, the two triangles overlap and we have

$$CKB + AHC > ACB \quad .$$

In summary

$$\text{sgn.}(\alpha+\beta-\gamma) = \text{sgn.}(CKB + AHC - ACB) \quad .$$

The three areas at the right-hand side are those of similar triangles and hence have the same ratios as the squares of corresponding lines, in particular

$$\frac{CKB}{a^2} = \frac{AHC}{b^2} = \frac{ACB}{c^2} > 0 \quad ;$$

hence

$$\text{sgn.}(CKB + AHC - ACB) = \text{sgn.}(a^2+b^2-c^2) \quad .$$

Hence we have proved

$$\text{sgn.}(\alpha+\beta-\gamma) = \text{sgn.}(a^2+b^2-c^2) \quad ,$$

a theorem, say, 4 times as rich as the one we quoted from Coxeter.

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The title of this note could make one wonder why I would waste my time flogging a horse as dead as Pythagoras's Theorem. So let us try to summarize what we could learn from this exercise.

• Three cheers for formalization! Instead of setting out to prove $a^2+b^2=c^2$ for a right-angled triangle, we included the antecedent $\gamma=\pi/2$ in the formal statement of what was to be proved. It was only after the introduction of π that we could eliminate it and met the "nicely symmetric" formulation.

• Three cheers for the equivalence! It makes quite clear that the theorem is not about right-angled triangles, but about triangles in general.

• Three cheers for the notational device captured in sgn. If we had not been careful, we would have ended up proving

$$(\alpha+\beta)\,\underline{R}\,\gamma \;\equiv\; (a^2+b^2)\,\underline{R}\,c^2$$

for $\underline{R}$ any of the six relations $=, \neq, <, \leq, >$, and $\geq$

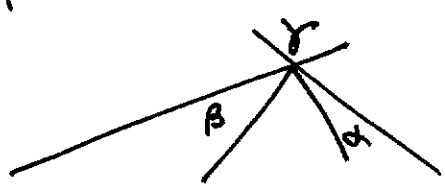
• No cheers at all for that stage of the argument in which lack of axiomatization forced us to resort to a picture. Pictures are almost unavoidably over-specific and thereby often force a case analysis upon you. Note that I carefully avoided the pictures for $\alpha+\beta>\gamma$; there are 9 of them: K to the right of A, coincident with A, and to the left of A, and similarly for the pair H and B. For the argument these distinctions are irrelevant but, when drawing a picture, you can hardly avoid making them.

• One of these days I would like to find a convincing explanation of the circumstance that youngsters continue to be educated with the theorem of Pythagoras in its diluted form as quoted from Coxeter. Notice that the 9 figures could have been avoided by also proving

$$CKB + AHC < ACB \Rightarrow \alpha + \beta < \gamma \quad \text{and}$$
$$CKB + AHC = ACB \Rightarrow \alpha + \beta = \gamma \quad ,$$

i.e. proving (0) and (1) in full.

• Notice that our figure was <u>not</u> pulled out of a magicians hat! As soon as $\mathrm{sgn.}(\alpha + \beta - \gamma)$ occurs in the demonstrandum, it is sweetly reasonable to construct that difference. In order not to destroy the symmetry between α and β, one starts with γ and subtracts α at the one side and β at the other:



and this is the germ of the figure we drew.

<u>Epilogue</u>. I am in a paradoxical situation. I am convinced that of the people knowing the theorem of Pythagoras, almost no one can read the above without being surprised at least once. Furthermore I think all those surprises relevant (because telling about their education in reasoning). Yet I don't know of a single respectable journal in which I could flog this dead horse.

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188, USA