A methodological remark on mathematical induction

This note is for the record: I know that I have shown the following during several lectures, but cannot remember that I ever wrote it down.

In the following, x and y range over the elements of a well-founded set $(C,<)$; $(C,<)$ being well-founded means that for any predicate P on C

(0) $\quad (\underline{A}x :: P.x) \equiv (\underline{A}x :: P'.x)$,

where P' is given in terms of P by

(1) $\quad (\underline{A}x :: P'.x \equiv P.x \vee (\underline{E}y : y<x : \neg P.y))$.

This is what mathematical induction is about: instead of computing the left-hand side of (0), we can compute its right-hand side, and, if the value is true, the latter computation is "easier" because - see (1) - P' is formally weaker than P. Indeed, we deduce from (1) immediately

(2) $\quad (\underline{A}x :: P.x \Rightarrow P'.x)$.

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The right-hand side of (0) is of the same form as its left-hand side, so, why don't we make life still easier by repeating the trick, since

$$(\underline{A}x :: P'.x) \equiv (\underline{A}x :: P''.x)$$

where according to (1) - with $P := P'$ -

(3) $\quad (\underline{A}x :: P''.x \equiv P'.x \vee (\underline{E}y : y<x : \neg P'.y))$.

Should we continue, and derive P''' to make life even easier? No, we can stop at P', for we observe for any x

$$\begin{array}{ll}
 & P''.x \\
= & \{(3)\} \\
 & P'.x \ \vee \ (\underline{E}y : y<x : \neg P'.y) \\
= & \{(1)\} \\
 & P.x \ \vee \ (\underline{E}y : y<x : \neg P.y) \ \vee \ (\underline{E}y : y<x : \neg P'.y) \\
= & \{\text{predicate calculus}\} \\
 & P.x \ \vee \ (\underline{E}y : y<x : \neg P.y \ \vee \neg P'.y) \\
= & \{(2) \text{ and predicate calculus}\} \\
 & P.x \ \vee \ (\underline{E}y : y<x : \neg P.y) \\
= & \{(1)\} \\
 & P'.x \quad .
\end{array}$$

In other words, the decision to demonstrate the truth of $(\underline{A}x :: P.x)$ by mathematical induction is idempotent: it can be taken once, but then the proof obligation has reached a fixpoint.

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