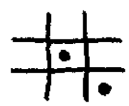
The checkers problem told to me by M.O.Rabin

Last week, during the annual seminar at the University of Newcastle-upon-Tyne, Michael O. Rabin told me the following problem.

Consider an infinite checkers board, of which the columns and rows are identified by the integer coordinates x and y respectively. Initially, there is a piece on each square whose coordinates satisfy $(\text{even}.x \equiv \text{even}.y) \wedge y \leq 0$. Pieces can be moved "upwards" by the usual "capturing moves": from  to  and from  to  .
The question posed is whether there is an upper bound on the y-coordinates of the squares that may be occupied.

* * *

Because the question asked is about y-coordinates, we abstract for a moment from the x-coordinate. The two original moves then become one and the same move: from  to  . In order to capture that a high piece is created from ("is as good as", "corresponds to") its two successors — those who know him see Fibonacci lurking around the corner! —, we give a piece (on a square) at height y a weight φ^y with φ

the positive root of $\varphi^2 = \varphi + 1$. The equation is chosen so that the weight of the new piece equals the weight of the two pieces it replaces, i.e. a move is neutral as far as total weight is concerned. By restricting ourselves to the positive root, we keep all weights positive, i.e. total weight a monotonic function of the number of pieces involved. Solving the equation yields

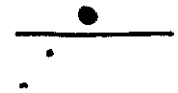
$$\varphi = (1 + \sqrt{5})/2 \quad .$$

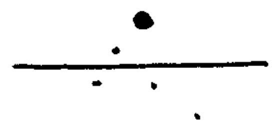
The fact that a move is neutral as far as total weight is concerned, means that the creation of a piece at height Y (with $Y > 0$) uses from the original configuration a set of pieces with total weight φ^Y . This "target weight", being φ^Y and φ being positive, grows exponentially with Y . We shall next observe that, in view of the shape of the two moves —x re-enters the picture—, the original pieces involved in the creation of a piece at height Y come from a restricted area. Calling their total weight the "available weight", we shall show that the latter grows linearly with Y. Hence, the condition

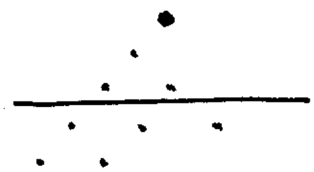
target weight $\leq$ available weight

imposes an upper bound on Y: the answer to the question posed is "Yes".

In order to establish the available weight we observe the moves for small values of Y .

$Y=1$ requires a single move, say 

$Y=2$ requires two moves, say  , the one thin dot above the line indicating a square that has been temporarily occupied

$Y=3$ requires two moves more. 

For the restricted areas from which the original pieces have to be recruited we take infinite (truncated) triangles

$Y=1$ $\qquad$ $Y=2$ $\qquad$ $Y=3$

etc. $\qquad$ etc $\qquad$ etc.

For $Y=1$, the available weight $(\Sigma n: n \geq 0: (n+1) * \varphi^{-n})$ is finite, and so is the increment $(\Sigma n: n \geq 0: \varphi^{-n})$. Summing the sequences one finds for the available weight

$$\frac{4+2\sqrt{5}+Y*(3+\sqrt{5})}{2}$$

The smallest value for Y such that the target weight φ^Y exceeds the available weight is 7 : then the target weight equals $(29+13\sqrt{5})/2$, the available weight is only $(25+9\sqrt{5})/2$.

For $Y=6$, the target weight $(18+8\sqrt{5})/2$ is less than the available weight $(22+8\sqrt{5})/2$, but this does not justify the conclusion that $Y=6$ is attainable. $Y=6$ can be achieved, but the only way of showing this that I know of is showing a game that does the job. The required game turns out to have 53 moves, which makes finding it and reporting it somewhat of a challenge. I could convince myself that $Y=6$ could be reached only

(i) after having realized that the constraint of at most 1 piece per square is inessential and can be dropped

(ii) after having decided to play the game backwards.

Below, we show successive stages of the backwards game: in the centre the configuration of the pieces, to the left the y-coordinates of the rows in question and –by way of check– to the right for each row the total number of pieces in it.

6					1		1
5				1			1
4			1				1
4			1		1		2
3						1	
3		1		1		1	3
2	1		1				2

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  2                 1   2   1   1                   5
  1                       1   1   1                 3

  1               1   1   2   2   2                 8
  0             1   1       1   1   1               5

  0             2   2   2   2   3   2              13
 -1           1   1   2   1   1   1   1             8

  0             1   1   1   1   1   1               6
 -1           2   2   3   2   2   2   2            15
 -2         1   1   1   1   1       1   1           7

  0             1   1   1   1   1   1               6
 -1           1   1   1   1   1   1   1             7
 -2         2   2   2   2   2   1   2   2          15
 -3       1   1   1       1   1   1   1   1         8
and now not repeating all the constant rows
 -1           1   1   1   1   1   1   1             7
 -2         1   1   1   1   1   1   1   1           8
 -3       2   1   2   1   2   2   1   2   2        15
 -4     1           1   1   1   1       1   1       7

 -2         1   1   1   1   1   1   1   1           8
 -3       1   1   1   1   1   1   1   1   1         9
 -4     2       1   1   2   2   1   1   1   2      13
 -5   1       1       1   1       1           1     6

 -3       1   1   1   1   1   1   1   1   1         9
 -4     1   0   1   1   1   1   1   1   1   1       9
 -5   1   1   1       2   1   1   1       1   1    10
 -6         1       1           1       1           4

 -4     1   0   1   1   1   1   1   1   1   1       9
 -5   1   1   1       1   1   1   1       1   1     9
 -6         1       1   1       1       1           5
 -7                       1                         1
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There may exist a game of 51 moves, but I am not interested in that optimization. The above is already more elaborate than I had hoped.

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