

It is all distributivity

Let  $\bullet$  be a binary infix operator. Depending on how we Curry, we can form two classes of unary operators from, the general elements of which are often denoted by  $(x\bullet)$  and  $(\bullet y)$ , defined

$$(x\bullet) = \langle \lambda y: x\bullet y \rangle \quad (\bullet y) = \langle \lambda x: x\bullet y \rangle .$$

We call them in this note "the derived unary operators".

To begin with we observe - with apologies for the two different usages of parentheses -

$$\begin{aligned} (x\bullet z)\bullet y &= x\bullet (z\bullet y) \\ &= \{ \text{definition of derived operators} \} \\ (\bullet y)\bullet (x\bullet z) &= (x\bullet)\bullet (\bullet y)\bullet z \\ &= \{ \text{definition of functional composition} \} \\ ((\bullet y)\bullet (x\bullet))\bullet z &= ((x\bullet)\bullet (\bullet y))\bullet z \end{aligned}$$

hence, that a binary operator is "associative" means that its derived unary operators commute.

To say that  $\bullet$  "distributes over" some binary operator is really a statement about its derived unary operators: " $\bullet$  distributes from the left over  $\square$ " means that for all  $x, y, z$ :

$$(x \circ). (y \square z) = (x \circ). y \square (x \circ). z ;$$

distribution from the right means similarly

$$(\bullet x). (y \square z) = (\bullet x). y \square (\bullet x). z .$$

Both formulae are of the form

$$f. (y \square z) = f.y \square f.z ,$$

which captures "f distributes over  $\square$ ". But what comes of this formula if  $\square$  turns out to be unary? Replacing in the last formula  $p \square q$  by  $g.p$ , we get

$$f.(g.y) = g.(f.y) \quad \text{or} \quad f \circ g = g \circ f .$$

Hence, that two unary operators "commute" means that they distribute over each other.

Hence, that a binary operator is associative just means that its derived unary operators distribute over each other.

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