The transitivity of the implication

Let $\sqsubseteq$ —read "below"— be a punctual relation on structures of some type, i.e.

(0) $[x=y \wedge u=z \Rightarrow (x \sqsubseteq u \equiv y \sqsubseteq z)]$.

(We have given $\sqsubseteq$ a higher binding power than the logical operators; universal quantification over free variables is left understood.)

Let $\sqsubseteq$ be transitive , i.e.

(1) $[x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z]$.

To the usual definition (1) , we prefer here the equivalent alternatives

(2) $[x \sqsubseteq y \Rightarrow (y \sqsubseteq z \Rightarrow x \sqsubseteq z)]$

and — with $x, y, z := z, x, y$ —

(3) $[x \sqsubseteq y \Rightarrow (z \sqsubseteq x \Rightarrow z \sqsubseteq y)]$.

Note that, if a relation is transitive, so is its transpose.

Punctual h is monotonic with respect to $\sqsubseteq$ means

(4) $[x \sqsubseteq y \Rightarrow h.x \sqsubseteq h.y]$;

punctual h is antimonotonic with respect to $\sqsubseteq$ means analogously

(5) $[x \sqsubseteq y \Rightarrow h.y \sqsubseteq h.x]$.

* * *

After all those general definitions, we now start with more specific observations. With functions f and g defined by

(6) $[f.\xi \equiv z \sqsubseteq \xi]$ and $[g.\xi \equiv \xi \sqsubseteq z]$ for all ξ ,

we can rewrite (2) and (3) :

(7) $[x \sqsubseteq y \Rightarrow (g.y \Rightarrow g.x)]$

(8) $[x \sqsubseteq y \Rightarrow (f.x \Rightarrow f.y)]$.

Comparing (7) with (5) —or (8) with (4)— we see that in the two positions where (5) —or (4)— has "$\sqsubseteq$", (7) —or (8)— has a "$\sqsubseteq$" and a "$\Rightarrow$" respectively. There is precisely <u>one</u> special case in which this difference disappears, viz. when for the punctual and transitive "$\sqsubseteq$" in (4) through (8) we choose the punctual and transitive "$\Rightarrow$". The choice gives for (2) and (3) respectively the well-known theorems

(9) $[(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))]$

—i.e. the implication is antimonotonic in its antecedent— and

(10) $[(x \Rightarrow y) \Rightarrow (z \Rightarrow x) \Rightarrow (z \Rightarrow y)]$

—i.e. the implication is monotonic in its consequent— .

And now we see why the implication is such a special relation: it is —with its transpose "$\Leftarrow$"— the <u>only</u> relation whose transitivity can be expressed in terms of (anti)monotonic dependence on its arguments! It is this circumstance that justifies and explains the urgent and wise advice W.H.J. Feijen gave me —more than once, as a matter of fact, for I was a little bit slow in the uptake— . The advice was to justify steps like

$$\begin{array}{l} \quad A \Rightarrow B \\ \Leftarrow \ \{\text{hint why } [A \Rightarrow C] \text{ and how to use this}\} \\ \quad C \Rightarrow B \end{array}$$

<u>not</u> in terms of transitivity of "$\Rightarrow$", but as strengthening by weakening the antecedent. To strengthen and to weaken we have to do anyhow; the elimination of implication's transitivity from the language of hints is a significant simplification.

Nuenen, 12 December 1992

prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA