A theorem proved by John Segers

On Tuesday 25 January 1994, the ETAC (= Eindhoven Tuesday Afternoon Club) was invited/challenged to prove the following theorem (that had been proved and communicated by John Segers)

(0) $\quad [\dagger\neg\dagger\neg\dagger y \equiv \neg\dagger\neg\dagger y]$ for all y,

where $\dagger$ (read: "dagger") denotes the non-reflexive transitive closure, i.e. $\dagger p$ denotes the strongest solution of, for instance: $\quad x: [p \vee x;x \Rightarrow x]$. We use

(1) $\quad [p \Rightarrow \dagger p]$

(2) $\quad [\dagger p ; \dagger p \Rightarrow \dagger p]$

(3) $\quad [p ; \dagger p \Rightarrow \dagger p]$

(4) $\quad [p \Rightarrow x] \wedge [p;x \Rightarrow x] \quad \Rightarrow \quad [\dagger p \Rightarrow x]$,

all for all p, x.

Note (3) follows from (1) and (2). (End of Note.)

The ETAC met the challenge. We give our proof before discussing it.

We begin by introducing b and c by

(5) $[b \equiv \neg \Psi \neg \Psi y]$, $[c \equiv \neg \Psi y]$, $[b = \neg \Psi c]$,

which allows us to rewrite our demonstrandum (0) as $[\Psi b \equiv b]$, for the proof of which we observe

$$
\begin{array}{ll}
 & [\Psi b \equiv b] \\
= & \quad \{\text{pred. calc.}\} \\
 & [\Psi b \Rightarrow b] \;\wedge\; [b \Rightarrow \Psi b] \\
= & \quad \{(1) \text{ with } p := b\} \\
 & [\Psi b \Rightarrow b] \\
\Leftarrow & \quad \{(4) \text{ with } p, x := b, b\} \\
 & [b \Rightarrow b] \;\wedge\; [b ; b \Rightarrow b] \\
= & \quad \{\text{pred. calc}\} \\
 & [b ; b \Rightarrow b] \\
= & \quad \{(5)\} \\
 & [\neg \Psi c ; \neg \Psi c \Rightarrow \neg \Psi c] \\
= & \quad \{\text{pred. calc}\} \\
 & [\Psi c \Rightarrow \neg(\neg \Psi c ; \neg \Psi c)] \\
\Leftarrow & \quad \{(4) \text{ with } p, x := c, \neg(\neg \Psi c ; \neg \Psi c)\} \\
 & [c \Rightarrow \neg(\neg \Psi c ; \neg \Psi c)] \;\wedge \\
 & [c ; \neg(\neg \Psi c ; \neg \Psi c) \Rightarrow \neg(\neg \Psi c ; \neg \Psi c)]
\end{array}
$$

We now deal with these two conjuncts separately. For the last conjunct we observe

$$
\begin{array}{ll}
 & [c ; \neg(\neg \Psi c ; \neg \Psi c) \Rightarrow \neg(\neg \Psi c ; \neg \Psi c)] \\
= & \quad \{\text{rel. calc., REX in particular}\} \\
 & [\sim c ; \neg \Psi c ; \neg \Psi c \Rightarrow \neg \Psi c ; \neg \Psi c] \\
\Leftarrow & \quad \{; \text{ is associative and monotonic}\} \\
 & [\sim c ; \neg \Psi c \Rightarrow \neg \Psi c]
\end{array}
$$

$$
\begin{array}{ll}
 & [\sim c \,;\, \neg\psi c \Rightarrow \neg\psi c] \\
= & \quad \{\text{REX}\} \\
 & [c \,;\, \psi c \Rightarrow \psi c] \\
= & \quad \{(3) \text{ with } p := c\} \\
 & true \quad ,
\end{array}
$$

so, the last conjunct has been proved, independent of the internal structure of c. We now proceed on the assumption that the proof of the first conjunct does require the internal structure of c as given in (5) : $[c \equiv \neg\psi y]$. That c is a negation of a predicate is irrelevant, for this holds for any predicate. However, $[\neg c \equiv \psi y]$, i.e. the fact that $\neg c$ is a transitive closure of y could be relevant. Since y <u>only</u> occurs in the combination ψy, we look among (1) through (4) for those properties of ψ that contain p <u>only</u> in the combination ψp, i.e. (2). We observe therefore

$$
\begin{array}{ll}
 & true \\
= & \quad \{(2) \text{ with } p := y\} \\
 & [\psi y \,;\, \psi y \Rightarrow \psi y] \\
= & \quad \{(5), \text{ i.e. } [\neg c \equiv \psi y]\} \\
(6) & [\neg c \,;\, \neg c \Rightarrow \neg c] \qquad ;
\end{array}
$$

in other words: if the internal structure of c plays a role, it has to do so via (6). We now observe for the first conjunct

$$
\begin{array}{ll}
 & [c \Rightarrow \neg(\neg\psi c ; \neg\psi c)] \\
= & \quad \{\text{pred. calc.}\} \\
 & [\neg\psi c ; \neg\psi c \Rightarrow \neg c] \\
\Leftarrow & \quad \{(6)\} \\
 & [\neg\psi c ; \neg\psi c \Rightarrow \neg c ; \neg c] \\
\Leftarrow & \quad \{; \text{ monotonic}\} \\
 & [\neg\psi c \Rightarrow \neg c] \\
= & \quad \{\text{pred. calc.}\} \\
 & [c \Rightarrow \psi c] \\
= & \quad \{(1) \text{ with } p := c\} \\
 & \mathit{true}
\end{array}
$$

And this concludes the proof.

* * *

The introduction of b is justified because the first 4 steps of the proof are independent of b's internal structure. At the ETAC session of 1994.01.25, we "pealed off" the operators one at a time, i.e. our next step was to introduce $[d \equiv \psi\neg\psi y]$ and eliminate b by $[b \equiv \neg d]$, but, though correct, I consider in retrospect the introduction of d overdone: admittedly, the introduction of d postpones the introduction of a few daggers but it is mathematically void since any b is the negation of something. We also used the "confront" – the conjugate of composition – because the subexpression $\neg(\neg d ; \neg d)$ asked for it, but in retrospect it was not worth the trouble.

When we performed the second step, viz. "{ ; is associative and monotonic }" in the proof of the second conjunct, we felt we were doing something absolutely standard: in our experience, an appeal to the associativity of composition is almost always indicated when a composition becomes an argument of a composition.

In the proof of the first conjunct, the appeal to (6) – which we had not formulated in advance – came as a surprise. I had not seen that (6) is the only way to exploit that Ψy is a transitive closure. (Note (x is transitive) ≡ (x is a transitive closure).)

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