# The transitive closure of a wellfounded relation

Transitive closures can be defined in many ways, but today we define the nonreflexive transitive closure $s$ of a relation $r$ as the strongest $s$ satisfying

(0) $[r \vee r;s \equiv s]$

From (0) alone – i.e. not using that $s$ is the strongest – we can derive

(1) $\langle \forall x :: [x \equiv s;x] \Rightarrow [x \equiv r;x] \rangle$

<u>Proof</u> We observe for an $x$ satisfying

(2) $[x \equiv s;x]$

$$
\begin{array}{ll}
 & x \\
\equiv & \{ (2) \} \\
 & s;x \\
\equiv & \{ (0) \} \\
 & (r \vee r;s);x \\
\equiv & \{ ; \text{ over } \vee \text{ and associative} \} \\
 & r;x \vee r;s;x \\
\equiv & \{ (2) \} \\
 & r;x \vee r;x \\
\equiv & \{ \text{pred. calc.} \} \\
 & r;x
\end{array}
$$

(End of Proof)

One of the formulations of "r is left-wellfounded" is

(3) $\langle \forall x :: [x \equiv r;x] \Rightarrow [\neg x] \rangle$ .

Thanks to (1), (3) implies

$$\langle \forall x :: [x \equiv s;x] \Rightarrow [\neg x] \rangle \quad ,$$

in other words: if a relation is left-wellfounded, so is its nonreflexive transitive closure.

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From (0) alone — i.e. not using that s is the strongest — we can derive

(4) $\langle \forall x :: [x \Rightarrow r;x] \Rightarrow [x \Rightarrow s;x] \rangle$

<u>Proof</u> We observe for any x

$[x \Rightarrow s;x]$

$\equiv$ { (0) }

$[x \Rightarrow (r \vee r;s);x]$

$\Leftarrow$ { pred. calc. and monotonicity of ; }

$[x \Rightarrow r;x]$

(End of Proof.)

Thanks to Knaster-Tarski, an alternative formulation of "s is left-wellfounded" is

(5) $\langle \forall x :: [x \Rightarrow s;x] \Rightarrow [\neg x] \rangle$ .

From (4) and (5) we derive

$$\langle \forall x :: [x \Rightarrow r;x] \Rightarrow [\neg x] \rangle \quad ,$$

in other words: if the nonreflexive transitive closure of a relation is left-wellfounded, so is the relation itself.

<u>Remark</u> It is worth noting that the proofs of the crucial implications (1) and (4) use neither wellfoundedness nor the fact that $s$ is the strongest $s$ satisfying (0). (End of Remark.)

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For left-wellfounded $r$, (0) determines $s$ uniquely, i.e. given

(6) $\quad [r \vee r;s \equiv s]$

(7) $\quad [r \vee r;t \equiv t]$

(8) $\quad \langle \forall x :: [x \Rightarrow r;x] \Rightarrow [\neg x] \rangle$

we have to prove $[s \equiv t]$

<u>Proof</u> For reasons of symmetry, it suffices to prove $[t \Rightarrow s]$. We observe

$\quad [t \Rightarrow s]$

$\equiv \quad$ {pred. calc.}

$\quad [\neg (t \wedge \neg s)]$

$\Leftarrow \quad$ {(8) with $x := t \wedge \neg s$}

$[t \wedge \neg s \Rightarrow r;(t \wedge \neg s)]$

$\equiv$ { shunting and (6)}

$[t \Rightarrow r \vee r;s \vee r;(t \wedge \neg s)]$

$\equiv$ { ; over $\vee$ and pred. calc.}

$[t \Rightarrow r \vee r;(t \vee s)]$

$\Leftarrow$ { monotonicity of ; }

$[t \Rightarrow r \vee r;t]$

$\equiv$ { (7)}

true

(End of Proof.)

The above is a considerable streamlining of AvG88/EWD1079 dd 28 April 1990 (which made no use of the relational calculus).

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