

The problem of the maximum length of an ascending subsequence.

We consider a sequence of N elements $A(1)$ through $A(N)$. The order of increasing subscript value will be denoted by "the order from left to right". From such a sequence we can take so-called "subsequences of length s " by the removal of an arbitrary collection of $N-s$ elements and retaining the remaining s elements in the order in which they occurred in the original sequence. When, in addition, each element has an integer value, we call a subsequence "ascending" iff it contains no element with a right-hand neighbour smaller than itself.

Note. According to this definition, all N subsequences of length 1 and even the empty subsequence are ascending ones. (End of note.)

We look for an algorithm that determines for any given sequence (with $N > 0$) the maximum length of an ascending subsequence that can be taken from it.

Note. Although there need not be a unique longest ascending subsequence, the maximum length is unique, e.g. $3\ 1\ 1\ 2\ 4\ 3$ gives 4 for the maximum length, realized either by $1\ 1\ 2\ 4$ or by $1\ 1\ 2\ 3$. (End of note.)

If k represents the value we are looking for, we seek to establish the relation

R: $k =$ the maximum length of an ascending subsequence taken from $A(1)$ through $A(N)$.

Because R contains the parameter N , it is strongly suggested to take as invariant relation --or, as we shall see in a moment: as part of the invariant relation--

$P1(k, n)$: $k =$ the maximum length of an ascending subsequence taken from $A(1)$ through $A(n)$.

It has the virtues that it would do the job in the sense that $(P1(k, n) \text{ and } n = N) \Rightarrow R$ and is easily established, e.g. by $k, n := 1, 1$. These observations suggest to establish $P1(k, n)$ for $n = 1$ and then to increase n under invariance of $P1(k, n)$ until $n = N$, more precisely: to increase n repeatedly by 1 and to restore each time, when destroyed, the truth of $P1(k, n)$ by adjusting the value of k . Because extension with a next element can never decrease the maximum length of an ascending subsequence and can increase it by at most 1, the adjustment of k , when

needed, will have the form $k := k + 1$. More precisely: because

$$P1(k, n) = wp("n := n + 1", P1(k, n - 1))$$

we have to investigate after " $n := n + 1$ " under which circumstances no adjustment of k is needed, i.e. when

$$P1(k, n - 1) \Rightarrow P1(k, n) \quad , \quad (1)$$

and under what circumstances adjustment of k is needed, i.e. when

$$P1(k, n - 1) \Rightarrow P1(k + 1, n) \quad . \quad (2)$$

Relation (2) holds iff $A(n)$ can be used to extend an ascending subsequence of maximum length ($= k$) taken from $A(1)$ through $A(n - 1)$; this is true iff

$$A(n) \geq \text{the smallest rightmost value of an ascending subsequence of length } k \text{ taken from } A(1) \text{ through } A(n - 1) .$$

This last inequality shows us, that besides k --as defined by $P1(k, n)$ -- we would also like to store the minimum rightmost value --let us call it m for a moment-- of an ascending subsequence of maximal length. If (2) holds, k is obviously to be adjusted by $k := k + 1$, and the assignment $m := A(n)$ would make m again equal to the minimum rightmost value of an ascending subsequence of maximal length (because, in this case, all ascending subsequences of maximal length taken from $A(1)$ through $A(n)$ will have $A(n)$ as their rightmost element.)

The introduction of m as the minimum value of the rightmost value of an ascending subsequence of length k , presents, however, a problem in case (1). In that case, the extension with $A(n)$, although not leading to an increase of k , may require adjustment of m as it may lead to a decrease of the minimum rightmost value of an ascending subsequence of that unchanged maximal length. This would be the case if the value $A(n)$ --which now satisfies $A(n) < m$ -- could be used to extend an ascending subsequence of length $k - 1$, taken from $A(1)$ through $A(n - 1)$. In order to decide that, we would also need the minimum rightmost value of an ascending subsequence of length $k - 1$. Repeating the argument, we conclude that, instead of a scalar m , we need in addition to k an array variable m satisfying

$$P2(k, n, m): \quad \text{for all } j \text{ satisfying } 1 \leq j \leq k \\ m(j) = \text{the minimum rightmost value of an ascending subsequence} \\ \text{of length } j \text{ and taken from } A(1) \text{ through } A(n) .$$

Our total invariant relation will be $P1$ and $P2$.

Again, for $n = 1$, P2 is easily initialized --with $m(1) = A(1)$ --; we have to investigate, however, what updating obligations for the array variable m are implied by our duty to keep P2 invariant. The crucial discovery in the analysis of our updating obligations for the array variable m is that the elements of m itself are ascending, more precisely:

$$(1 \leq i < j \leq k) \Rightarrow (m(i) \leq m(j)) \quad .$$

This follows from the fact that $1 \leq i < j \leq k$ and $m(i) > m(j)$ leads to a contradiction: by removing from an ascending sequence of length j and with $m(j)$ as its rightmost value the leftmost $j-i$ elements, an ascending sequence of length i with $m(j)$ as rightmost value remains, and $m(i) > m(j)$ then contradicts P2 .

Again we investigate the situation as reached after $n := n + 1$, i.e. when $P1(k, n - 1)$ and $P2(k, n - 1, m)$ holds. Relation (2) holds iff $A(n) \geq m(k)$. The new element $A(n)$ can be used to form a longer ascending sequence, k has to be increased and the sequence of values is extended with $A(n)$ by

$$m:\text{hiext}(A(n)) \quad ;$$

it is correct to leave the values $m(i)$ with $1 \leq i < k$ unchanged, for the new element $A(n) \geq m(k)$ and can never give rise to a smaller rightmost value for any of the ascending sequences shorter than the new maximum length k . Relation (1) holds iff $A(n) < m(k)$. Remembering that after the increase $n := n + 1$ the relation $P2(k, n - 1, m)$ holds, we have to answer the question: for which value(s) of j is the minimum rightmost value of an ascending sequence of length j take from $A(1)$ through $A(n)$ smaller than taken from $A(1)$ through $A(n-1)$? This can only be the case if $A(n)$ is its new rightmost element, which must be smaller than its old value $m(j)$. So we have

$$A(n) < m(j) \tag{3}$$

But $A(n)$ can only be the rightmost element of an ascending sequence of length j if

$$\text{either } j = 1 \text{ or } j > 1 \text{ and } m(j-1) \leq A(n) \tag{4}$$

Combining (3) and (4) we find

$j = 1$ iff $A(n) < m(1)$ and otherwise $j =$ the only(!) solution of $m(j-1) \leq A(n) < m(j)$. This last solution is found in the program with a binary search; the invariant relation for the inner loop is $m(i) \leq A(n) < m(j)$.

Observing that $k = m.hib$, the current higher bound for the index, we can use for $m(k) = m(m.hib)$ the usual abbreviation $m.high$ and conclude that we don't need the variable k after all. For reasons of symmetry we denote $m(1)$ by $m.low$, as $m.lob = 1$. Omitting all declarations we get the following program.

```

n:= 1; m:= (1, A(1));
do n  $\neq$  N  $\rightarrow$ 
    n:= n + 1;
    if A(n)  $\geq$  m.high  $\rightarrow$ 
        m:hiext(A(n))
    [] A(n) < m.high  $\rightarrow$ 
        if m.low > A(n)  $\rightarrow$ 
            j:= 1
            [] m.low  $\leq$  A(n)  $\rightarrow$ 
                i, j := m.lob, m.hib;
            do i  $\neq$  j - 1  $\rightarrow$ 
                h:= (i + j)div 2;
                if m(h)  $\leq$  A(n)  $\rightarrow$  i:= h
                [] A(n) < m(h)  $\rightarrow$  j:= h
            fi
        od
    fi;
    m:(j)= A(n)
fi
od;
print(m.hib)

```

Plataanstraat 5
 NL-4565 NUENEN
 The Netherlands

prof.dr.Edsger W.Dijkstra
 Burroughs Research Fellow