On weak and strong termination.

In the literature we find two concepts of "termination"; we shall call them "weak termination" and "strong termination" respectively, equivalent within the realm of continuous functions, but different in the presence of unbounded nondeterminacy. It will be shown that in the realm of continuous functions the generality of (infinite) well-founded sets is of no essential use for proofs of termination, as partially ordered finite sets will do just as nicely.

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In a proof of weak termination we demonstrate the impossibility that a computation will continue "forever", although an upper bound on the "time" it will take need not exist; in a proof of strong termination we demonstrate that the computation will have terminated within a certain amount of "time".

For proofs of strong termination the conceptually simplest tool is the so-called "variant function", an integer-valued function of the state which is bounded from below (≥ 0 , say), and decreased by at least 1 at each "step" of the computation.

For proofs of weak termination Floyd [1967] has suggested to replace, as range of the variant function, the natural numbers by the elements of a so-called "well-founded set". A well-founded set is a set on which a (partial) ordering has been defined such that no element is the first of an infinite decreasing sequence of elements from the set. A well-known example of a well-founded set is the one consisting of the pairs (x,y) of natural numbers with the ordering defined as

$$(x',y') < (x,y) \underset{\text{def}}{=} x' < x \ \underline{\text{or}}\ (x' = x \ \underline{\text{and}}\ y' < y) \quad .$$

This well-founded set would be the proper vehicle for proving the weak termination of -- X and Y being natural constants--

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S:      x, y := X, Y;
        do x > 0 → x, y := x-1, any natural number
         [] y > 0 → y:= y-1
        od
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where "any natural number" denotes a function of unbounded nondeterminacy, i.e. such that

$$wp(\text{"y:= any natural number"}, y \geq 0) = T \quad \text{and}$$
$$wp(\text{"y:= any natural number"}, y \leq k) = F \quad \text{for } \underline{\text{all}} \ k \ .$$

Note that in general program S does not enjoy the property of strong termination, because for $X > 0$ no upper bound for y can be given.

The well-founded set of the pairs (x,y) used above nicely illustrates the way in which well-founded sets are a true generalization of the natural numbers. Each natural number n is the first element of only finite decreasing sequences, but only of a finite number of them -- 2^n, to be precise-- that, therefore, have a maximum length -- $n+1$, to be precise-- . In the more general well-founded set we considered, each element (x,y) with $x \geq 1$ is the first element of only finite decreasing sequences, but of infinitely many of them, whose lengths have no maximum. Our example also suggests that the generality the well-founded sets offer over and above the natural numbers is the last thing we need.

With program S we showed how, under assumption of the availability of the function "any natural number" of unbounded nondeterminacy, we could implement a weakly terminating program that was not strongly terminating. On the other hand it is quite easy to derive from <u>any</u> weakly terminating program that does not terminate strongly a computation of "any natural number": just add to it a count of the number of "steps" executed. Therefore the availability of the function "any natural number" of unbounded nondeterminacy is equivalent to the existence of programs that terminate weakly, but not strongly. Furthermore it is known --see, for instance, Dijkstra [1976], Chapter 9 -- that unbounded nondeterminacy is incompatible with the constraint of continuity.

Several conclusions present themselves:

1) Within the realm of continuous functions, where nondeterminacy is bounded, weak termination and strong termination are equivalent.

2) We only need the greater generality of the well-founded sets over and above the natural numbers, when we decide to leave the realm of the continuous functions. As long as there is very little incentive to do so,

that greater generality of (infinite) well-founded sets is of no essential use, and (partially) ordered finite sets will do just as nicely. (As a partial order on a finite set can always be embedded in a total order, the prevalence of the use of the range of natural numbers --the first K , for some sufficiently large K , to be precise-- becomes now fully understandable.)

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Floyd, R.W. [1967] "Assigning Meanings to Programs". Proc. Symp. in Applied Mathematics, vol. 19 (J.T.Schwartz, ed.), American Mathematical Society, Providence, RI, U.S.A.

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