On W.H.J. Feijen's solution for the lexicographic minimum of a circular list.

For a given, non-empty — i.e. with $N \geq 1$ — list

$$(C[0], C[1], \ldots, C[N-1])$$

we consider the N associated sequences C_i

$$C_i = (C[i], C[i+1], \ldots, C[i+N-1])$$

in which all subscripts are reduced modulo N. We are asked to design a program determining an integer value k satisfying the relation

$$R: \quad 0 \leq k < N \text{ and } (\forall i: 0 \leq i < N: C_i \not\prec C_k) \quad ,$$

in which "$\succ$" should be read as "lexicographically greater than".

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The lexicographic minimum CC is defined by the following two relations

$$(\forall i: 0 \leq i < N: C_i \not\prec CC) \qquad (0)$$
$$(\exists i: 0 \leq i < N: C_i = CC) \qquad (1)$$

From the transitivity of the lexicographic ordering we conclude

$$(\forall p,q :: C_p \not\prec C_q \text{ or } C_q \succ CC) \qquad (3)$$

<u>Note</u>. When, in a quantification, the range is obvious, we allow ourselves its omission. Hence the "::". (End of Note.)

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Let $P(k,n)$ be defined as:

$$P:\quad 0 \leq k < n \ \underline{\text{and}}\ (\forall i: 0 \leq i < n: C_i \succ CC \ \underline{\text{or}}\ i = k) \qquad (4)$$

We observe immediately

$$(P \ \underline{\text{and}}\ n \geq N) \Rightarrow R \qquad (5)$$

$$(\forall i: 0 \leq i < h: C_i \succ CC) \Rightarrow P(h, h+1) \qquad (6)$$

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Let $Q(k,n,\ell)$ be defined as

$$Q:\ 0 \leq k < n \wedge \ell \geq 0 \wedge (\forall i: 0 \leq i < \ell: C[k+i] = C[n+i]) \qquad (7)$$

Because the lexicographic order of two sequences with equal leading elements does not change when those leading elements are removed from them, we conclude

$$Q(k,n,\ell) \Rightarrow (Qa \ \underline{\text{or}}\ Qb \ \underline{\text{or}}\ Qc) \qquad (8)$$

with the mutually exclusive terms

$$Qa:\quad (\forall i: 0 \leq i < N: C_{k+i} = C_{n+i}) \qquad (8a)$$

$$Qb:\quad (\forall i: 0 \leq i \leq \ell: C_{k+i} \prec C_{n+i}) \wedge \ell \geq 0 \qquad (8b)$$

$$Qc:\quad (\forall i: 0 \leq i \leq \ell): C_{k+i} \succ C_{n+i}) \wedge \ell \geq 0 \qquad (8c).$$

We furthermore observe that, for $\ell = 0$, the last term of $Q(k,n,\ell)$ is vacuously true.

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We observe that $P \wedge Q$ can be initialized by

$$k, n, \ell := 0, 1, 0$$

and is left invariant by the following three guarded commands.

1) $C[k+\ell] = C[n+\ell] \rightarrow \ell := \ell+1$

This follows immediately from (4) — ℓ does not occur in P — and (7)

2) $C[k+\ell] < C[n+\ell] \rightarrow \{P \wedge Qb\}\ n, \ell := n+\ell+1, 0$

Q and the guard imply together Qb, and note that Qb implies

$$(\forall i: n \leq i < n+\ell+1: C_i \succ CC).$$

Hence, on account of (4), the invariance of P is guaranteed. The assignment $\ell := 0$ ensures the invariance of Q

3) $C[k+\ell] > C[n+\ell] \rightarrow \{P \wedge Qc\}$

$$h := max(n, k+\ell+1);$$
$$k, n, \ell := h, h+1, 0$$

Q and the guard imply together Qc, and note that Qc implies

$$(\forall i: k \leq i < k+\ell+1: C_i \succ CC) \wedge \ell \geq 0.$$

With $h = max(n, k+\ell+1)$ we conclude from the last result and P

$$(\forall i: 0 \leq i < h: C_i \succ CC)$$

and, thanks to (6), the invariance of P is guaranteed. The assignment $\ell := 0$ ensures the invariance of Q.

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Finally we show that

$$(P \wedge Q \wedge k+\ell+1 \geq N) \Rightarrow R \qquad (9)$$

1) $P \wedge Qa \wedge k+\ell+1 \geq N$

From Qa we conclude that the sequence is periodic, and that the period divides $n-k$; hence

$$(\exists i : k \leq i < n : C_i = CC)$$

Together with - what follows from P -

$$(\forall i : k < i < n : C_i \succ CC)$$

we conclude $C_k = CC$, i.e. R.

2) $P \wedge Qb \wedge k+\ell+1 \geq N$

From Qb we conclude, as before,

$$(\forall i : n \leq i < n+\ell+1 : C_i \succ CC) \quad ;$$

because $n > k$, $n+\ell+1 > k+\ell+1$; because $k+\ell+1 \geq N$ we conclude, together with P

$$(\forall i : 0 \leq i < N : C_i \succ CC \text{ \underline{or} } i = k) \quad , \text{i.e. } R$$

3) $P \wedge Qc \wedge k+\ell+1 \geq N$

From Qc we conclude, as before

$$(\forall i : k \leq i < k+\ell+1 : C_i \succ CC) \quad ;$$

because $k+\ell+1 \geq N$, we conclude together with P

$$(\forall i : 0 \leq i < N : C_i \succ CC) \quad ;$$

this, together with (1) implies <u>false</u>, and <u>false</u> $\Rightarrow R$.

(End of Proof of (9).)

Using (5) and (9), we see that the following program establishes R

$k, n, \ell := 0, 1, 0$ $\{P \wedge Q\}$;
<u>do</u> $n < N$ <u>and</u> $k + \ell + 1 < N \rightarrow$
  <u>if</u> $C[k+\ell] = C[n+\ell] \rightarrow \ell := \ell + 1$ $\{P \wedge Q\}$
  ▯ $C[k+\ell] < C[n+\ell] \rightarrow n, \ell := n + \ell + 1, 0$ $\{P \wedge Q\}$
  ▯ $C[k+\ell] > C[n+\ell] \rightarrow h := \max(n, k+\ell+1)$;
    $k, n, \ell := h, h+1, 0$ $\{P \wedge Q\}$
  <u>fi</u> $\{P \wedge Q\}$
<u>od</u> $\{R\}$

<u>Termination</u>. The function $t = k + n + \ell$ increases by at least one at each iteration. Before the first iteration $t = 1$, before the last one $t \leq 2N-3$, hence $2N-3$ is an upper bound for the number of comparisons.

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The above is -or will be- described much more beautifully by W.H.J. Feyen, including all the heuristics and acknowledgements. I wrote the above as part of my personal correspondence, for the sake of quick dissemination; I owe everything of the above to ir. W. H. J. Feijen.

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