On two types of infinite sets of infinite sequences.

by the Tuesday Afternoon Club.

In the following $x$ will stand for the infinite sequence

$$(x(0),\ x(1),\ x(2),\ \ldots\ldots\ldots\ldots)$$

where the $x(i)$ are taken from a finite alphabet of at least two characters, say $\{0, 1\}$ .

Let us consider the sets of sequences $S1$ and $S2$ , defined as the solution sets of the equations $P1(x)$ and $P2(x)$, respectively, with

$P1(x)$: $(\underline{E}\ i: i \geq 0: (\underline{A}\ j: 0 \leq j < i: x(j) = 0)\ \underline{\text{and}}\ (\underline{A}\ j: i \leq j: x(j) = 1))$

$P2(x)$: $(\underline{A}\ i: i \geq 0: x(i) = 0\ \underline{\text{or}}\ (\underline{A}\ j: j \geq i: x(j) = 1))$ .

Because $P1(x) \Rightarrow P2(x)$ , $S1$ is a subset of $S2$ , even a proper subset: there exists one solution of $P2(x)$ <u>and</u> <u>non</u> $P1(x)$ , viz. the sequence with $x(i) = 0$ for all $i \geq 0$ .

The set $S2$ has the property that for any non-member of $S2$ --i.e. any solution of <u>non</u> $P2(x)$-- non-membership can be established on account of an initial segment of it. As a matter of fact, in this particular case only two elements suffice for this evidence, as follows from <u>non</u> $P2(x)$:

$$(\underline{E}\ i: i \geq 0: x(i) \neq 0\ \underline{\text{and}}\ (\underline{E}\ j: j \geq i: x(j) \neq 1))\ .$$

The set $S1$ doesn't have this property because there exists a non-member --viz. $x(i) = 0$ for all $i \geq 0$-- with the property that any initial segment of it is also the initial segment of a member of $S1$ . We call $S2$ "closed" and $S1$ "non-closed", that is:

"The set $S$ is closed" means "any non-member of $S$ has an initial segment that is not the initial segment of any member of $S$".

"The set $S$ is non-closed" means "there exists a non-member of $S$ such that any initial segment of it is also the initial segment of some member of $S$ ".

From the above it follows

1) that a set of sequences is either closed or non-closed,

2) that any finite set --in particular the empty set-- is closed,

3) that the (uncountably infinite) universe of all possible sequences is closed.

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The notion of a closed set is of significance in connection with non-deterministic infinite computations. Consider the set of possible output sequences y corresponding to

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"initialize;  i:= 0;                                          (1)
 do true → compute; print(y(i)); i:= i + 1 od"
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Here "initialize" and "compute" stand for terminating computations not affecting the value of i . If "initialize" and "compute" stand for deterministic computations, the set of possible output sequences consists of a single element; if "initialize" and (in particular) "compute" are non-deterministic, the corresponding set of possible output sequences can be infinite.

> "Program (1) is a continuous machine" means "in program (1) "initialize" and "compute" are both of bounded non-determinacy".

We can now prove the following

<u>Theorem 1</u>. The possible output sequences of program (1) form a closed set <u>or</u> program (1) is not a continuous machine.

<u>Proof</u>. Let S be the set of possible output sequences of program (1). Either S is closed --in which case Theorem 1 holds-- or S is non-closed. In the latter case, let x be a non-member of S such that each initial segment of x is also an initial segment of some member of S . Consider now the following program:

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"initialize;  i:= 0;                                          (2)
 do "y(0),...,y(i-1)  is an initial segment of  x " →
         compute; print(y(i)); i:= i + 1
 od; print(i)"    .
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Non-termination of (2) is excluded because $x$ is not a member of $S$. Because _each_ initial segment of $x$ is also the initial segment of some member of $S$, the final value of $i$ is unbounded. Hence, program (2) is a (weakly) terminating program of unbounded non-determinacy; hence "initialize" or "compute" is of unbounded non-determinacy, i.e. program (1) is not a continuous machine. (End of Proof.)

_Theorem 2_. A set of sequences is non-closed _or_ is the set of possible output sequences of some continuous machine.

_Proof_. Let $S$ be the set of sequences. Either set $S$ is non-closed --in which case Theorem 2 holds-- or it is closed. In the latter case consider program (1) with "initialize" deterministic and "compute" only constrained by the requirement that the next $y(i)$ leads to an initial segment of some element of $S$. Because the alphabet is finite, "compute" need not be of unbounded non-determinacy. By virtue of its construction program (1) may then generate _any_ member of $S$, and, $S$ being closed, it cannot generate any non-member of $S$. (End of Proof.)

Having identified closed sets of possible output sequences with continuous machines, we propose to ignore for the time being non-deterministic infinite computations to which non-closed sets correspond.

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