The analysis of a two-person game.

As long as possible, two players make a move in turn; the player faced with a situation in which no move is possible has lost the game.

The situation is given by two natural numbers (i.e. integers ≥ 0). A move being possible means that both numbers are positive; a move then consists in decreasing one of the numbers one or more times by the other, but such that the number decreased remains ≥ 0.

It is asked to characterize those initial situations in which the player who has the first move can guarantee to win.

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Our first observation is that, since no move affects the greatest common divisor of the two numbers, only their ratio matters. Our second observation is that, since the situation is symmetric in the two numbers, we can confine ourselves when moves are possible to a ratio ≥ 1. Our third observation is that, to start with, a move consists in decreasing the ratio by an integer; if then $0 < \text{ratio} < 1$, the rôles of smallest and largest number have been interchanged, and we should replace the ratio by its inverse. And this suggests to characterize the ratio by its finite expansion as continued fraction, more precisely by the sequence of positive integers

(0) $k_0, k_1, \ldots, k_n$

such that the ratio equals r_n, given by

$r_0 = k_0$ and $r_{i+1} = k_{i+1} + 1/r_i$.

A move with a non-maximal decrease decreases k_n by an amount $< k_n$; a move with a maximal decrease decreases n by 1 ("drops k_n").

The state $n=0 \wedge k_0=1$ corresponds to a ratio $=1$, which is a winning situation for the initial player. It is exceptional in the sense that all ratios >1 can be represented with $k_0 \geq 2$. For such states we define

$$\text{trail} = n - (\text{MAX } i: 0 \leq i \leq n \wedge k_i \geq 2 : i)$$

i.e. the length of the train of ones on which (0) ends. Handing your opponent a situation with trail odd forces him to return a situation with trail even. With trail even you can either hand your opponent a situation with trail odd or terminate the game and win. Hence trail even characterizes the remaining winning situations, i.e. the ratio $> (1+\sqrt{5})/2$.

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