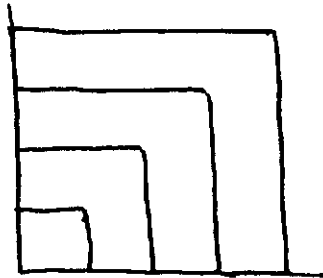


## Generalizing an old formula

It all started with a discussion about the role of pictures as substitute for calculational proofs. I was reminded of the pictorial illustration of the theorem that the sum of the first  $n$  odd natural numbers equals  $n^2$ :



Immediately the question rose whether this formula can be generalized to higher powers of  $n$ , since from the 4<sup>th</sup> power onwards, visualization breaks down.

By mentally cutting 3 mutually orthogonal slices of thickness 1 from a cube of  $n$  by  $n$  by  $n$ , one can convince oneself that

$$n^3 = \left( \sum_{i: 1 \leq i \leq n} i^2 + i \cdot (i-1) + (i-1)^2 \right)$$

and one begins to suspect the validity of

$$n^4 = \left( \sum_{i: 1 \leq i \leq n} i^3 + i^2 \cdot (i-1) + i \cdot (i-1)^2 + (i-1)^3 \right)$$

and similar formula, in general

$$n^k = \left( \sum_{i: 1 \leq i \leq n} \left( \sum_{j: 0 \leq j < k} i^{k-1-j} \cdot (i-1)^j \right) \right). \quad (0)$$

This holds provided

$$n^k = \left( \sum_{j: 0 \leq j < k} n^{k-1-j} \cdot (n-1)^j \right) + (n-1)^k \quad (1)$$

because (0) holds for  $n=0$ , while (1) then provides the step for proving (0) by induction over  $n$ .

We prove (1), which holds for  $k=1$ , by mathematical induction over  $k$ :

$$\begin{aligned}
 n^k &= \left( \sum_{j: 0 \leq j < k} n^{k-1-j} \cdot (n-1)^j \right) + (n-1)^k \\
 &\equiv \{ \text{multiply both sides by } n; n = 1 + (n-1) \} \\
 n^{k+1} &= \left( \sum_{j: 0 \leq j < k} n^{k-j} \cdot (n-1)^j \right) + (n-1)^k + (n-1)^{k+1} \\
 &\equiv \{ \text{definition of summation and } n^0 = 1 \} \\
 n^{k+1} &= \left( \sum_{j: 0 \leq j < k+1} n^{k+1-1-j} \cdot (n-1)^j \right) + (n-1)^{k+1} \\
 &\qquad\qquad\qquad \text{q.e.d.}
 \end{aligned}$$

The above was just for the record. I find the above double induction satisfactory and formula (0) was new for me.

Observation. In the mean time I asked people from five different countries: none of them had seen the above picture at school. We all agreed that that was a pity. (End of Observation)

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