Ptolemaeus and Brahmagupta (or: Baffled by Symmetry)

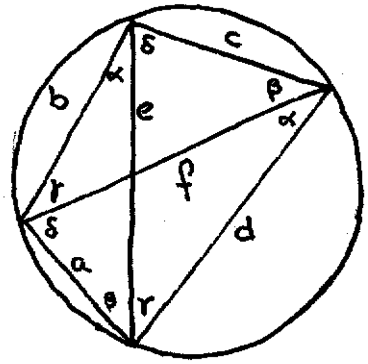

fig.0

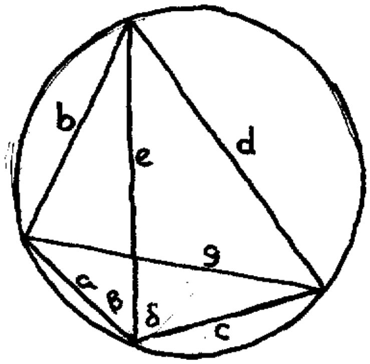

fig.1

We call a convex 4-gon whose vertices lie on a circle a "circular quadrangle", and consider circular quadrangles with sides a, b, c and d. There are essentially – i.e. apart from mirror images – 3 such circular quadrangles, with sides in order (a,b,c,d) – fig.0 – , (a,b,d,c) – fig.1 – and (a,c,b,d) – no figure – . Their diagonals are (e,f), (e,g) and (f,g); they have circumscribed circles with the same diameter D and have the same area A.

Furthermore we have with positive angles α, β, γ, and δ, such that $\alpha+\beta+\gamma+\delta = \pi$

$a = D\cdot \sin.\alpha \quad b = D\cdot \sin.\beta \quad c = D\cdot \sin.\gamma \quad d = D\cdot \sin.\delta$
$e = D\cdot \sin.(\alpha+\beta) \quad f = D\cdot \sin.(\beta+\gamma) \quad g = D\cdot \sin.(\beta+\delta)$

(Note that $\sin.(\alpha+\beta) = \sin.(\gamma+\delta)$ etc.)

Because twice the area of a convex 4-gon equals the product of the diagonals times the sine of the angle between them, we derive from fig. 0

$$2 \cdot A = e \cdot f \cdot \sin.(\beta + \delta) \quad . \qquad (0)$$

Because twice the area of a triangle equals the product of two sides times the sine of the angle between them, we derive from fig. 1

$$2 \cdot A = (a \cdot c + b \cdot d) \cdot \sin.(\beta + \delta) \quad . \qquad (1)$$

Hence

$$e \cdot f = a \cdot c + b \cdot d \quad ,$$

the famous theorem of Ptolemaeus. [For reasons of symmetry we also have

$$e \cdot g = a \cdot d + b \cdot c \quad \text{and} \quad f \cdot g = a \cdot b + c \cdot d \quad .]$$

From (0) and $g = D \cdot \sin.(\beta + \delta)$ we derive

$$2 \cdot A \cdot D = e \cdot f \cdot g \qquad (2)$$

(something I was not aware of).

The above proof of the theorem of Ptolemaeus is much nicer than the tricky one I was shown in my schooldays (which was the same one my parents had grown up with: even ugly arguments have a long life, once they have penetrated the school curriculum!). Yet I am annoyed by it because I compared fig. 0 with fig. 1, from which f has disappeared, whereas I could also have compared it with the omitted figure in which diagonal e has disappeared. I have been overspecific in a way I could not avoid.

An alternative proof of the theorem of Ptolemaeus shows that for

$$\alpha + \beta + \gamma + \delta = \pi \qquad (3)$$

we have

$$\sin.(\alpha+\beta)\cdot \sin.(\beta+\gamma) = \sin.\alpha \cdot \sin.\gamma + \sin.\beta \cdot \sin.\delta \qquad .(4)$$

Here the formulation of (the left-hand side of) (4) is arbitrary; the simplest computation I can think of makes it even worse. We can use (3) by eliminating δ from (4), i.e. for arbitrary α, β, and γ we have to show

$$\sin.(\alpha+\beta)\cdot \sin(\beta+\gamma) = \sin.\alpha \cdot \sin.\gamma + \sin.\beta \cdot \sin.(\alpha+\beta+\gamma) \quad .(5)$$

Relation (5) holds for $\beta = 0$; furthermore, the derivatives of both sides with respect to β are equal, viz. to $\sin.(\alpha + 2\cdot\beta + \gamma)$. Hence, (5) holds for any value of α, β, and γ . But as far as destruction of symmetry is concerned, this is even worse.

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With the theorem of Brahmagupta I have had similar experiences; it states

$$A^2 = (s-a)\cdot(s-b)\cdot(s-c)\cdot(s-d) \qquad (6).$$
$$\text{where} \quad s = (a+b+c+d)/2$$

The least cumbersome argument I know of
(i) establishes that A^2 is a homogeneous polynomial of degree 4 in a, b, c, and d ;
(ii) establishes that the right-hand side is a factor of A^2, i.e. A^2 = a constant times that right-hand side;
(iii) establishes that the constant equals 1 (e.g. by inspection of a square) .

Step (ii) follows from

$$s - a = 0 \quad \equiv \quad a = b + c + d \qquad ;$$

since the right-hand side implies $A^2 = 0$, $s - a$ is a factor, and so for the others. Step (i) is the problem.

With the cosine rule we derive from Fig. 1

$$g^2 = a^2 + c^2 - 2\cdot a\cdot c\cdot \cos.(\beta+\delta)$$
$$g^2 = b^2 + d^2 + 2\cdot b\cdot d\cdot \cos.(\beta+\delta) \qquad ;$$

hence, eliminating g^2, we find

$$(a\cdot c + b\cdot d)\cdot \cos.(\beta+\delta) = 2^{nd} \text{ degree polynomial} \quad .$$

Eliminating $\beta+\delta$ from the above and (1) by squaring both sides of both, one establishes the conclusion of (i).

Finally, one establishes (iii) by looking at the unit square.

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Needless to say, the above proof of Brahmagupta's Theorem - though the least cumbersome I have ever seen — still annoys me. It is a little bit less ugly than the proof of the Theorem of Ptolemaeus in that the demonstration of (ii) uses only fig.1 . But, firstly, that choice is unattractively arbitrary, and, secondly, we then made the arbitrary choice of focussing our attention on diagonal g .

My stumbling block is probably the algebraic identity

$$(a\cdot b + c\cdot d)\cdot(a\cdot c + b\cdot d)\cdot(a\cdot d + b\cdot c) =$$
$$a^3\cdot b\cdot c\cdot d + b^3\cdot c\cdot d\cdot a + c^3\cdot d\cdot a\cdot b + d^3\cdot a\cdot b\cdot c +$$
$$a^2\cdot b^2\cdot c^2 + b^2\cdot c^2\cdot d^2 + c^2\cdot d^2\cdot a^2 + d^2\cdot a^2\cdot b^2$$

the one side showing a 3-fold symmetry, the other one a 4-fold one.

Austin 24 August 1986

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