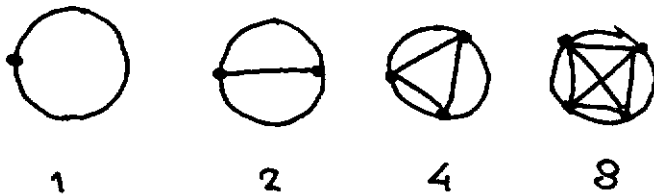


A solution designed by A. Blokhuis

Consider all the chords between  $n$  distinct points on the perimeter of a circle, the points having been chosen such that each point inside the circle lies on at most 2 chords. Into how many regions is the area inside the circle divided by those chords?

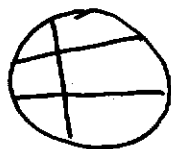


The guess that for  $n=5$  the number equals 16 is correct; for  $n=6$ , the number equals 31. (It is a very convincing example of the inadequacy of what is known as "Poor Man's Induction".)

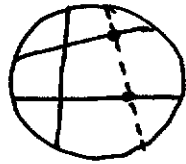
\* \* \*

The following argument derives an expression for the number of regions as function of  $n$ .

In an effort of disentanglement we try not to use all our data simultaneously, in particular we temporarily ignore that eventually we will deal with all chords between  $n$  points. What can we say about the number of regions when a number of chords has been drawn?



Let us draw another chord:



Let  $R$  = the number of regions  
 $C$  = the number of chords  
 $I$  = the number of intersections inside the circle  
 and let us investigate the relation between their increments for the case

$$\Delta C = 1 ;$$

$\Delta R$  is the number of segments into which the new chord is divided by the others, which is one more than the number of its intersections, i.e.

$$\Delta R = 1 + \Delta I ;$$

from the above we derive

$$\Delta R = \Delta C + \Delta I$$

and, since  $I=0$  and  $C=0$  corresponds to  $R=1$ ,  
 by induction over  $C$

$$R = 1 + C + I$$

So we are done if we can determine  $C$  and  $I$ .

This is the moment to remember that we were considering  $n$  "peripheral" points and all chords between them.

Each chord determines two distinct peripheral points, and vice versa, hence  $C = \binom{n}{2}$ .

Each intersection, by determining two distinct chords, determines 4 distinct peripheral points.

and vice versa \*\*) , hence  $I = \binom{n}{4}$

\*) Intersecting each other already inside the circle, the two chords don't share a peripheral point.

\*\*) For any 4 peripheral points, only 1 of the 3 possible pairings leads to chords intersecting inside the circle.

$$\text{So } R = 1 + \binom{n}{2} + \binom{n}{4}$$

Since  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$  , we can rewrite

$$R = 1 + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4} ,$$

i.e. the sum of the first 5 entries of a line of the triangle of Pascal. The full lines adding up to the successive powers of 2 , we now understand the misleading observations with which this note started.

The above argument is due to A. Blokhuis .

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