A summary for Turner's class (An extract from EWD1107)

Let $\sqsubseteq$ -read "below"- be a punctual relation on structures of some type, i.e. for all u, v, x, y

$$[\, u = v \ \wedge\ x = y \ \Rightarrow\ (u \sqsubseteq x \ \equiv\ v \sqsubseteq y)\,] \quad .$$

For $\sqsubseteq$ a preorder -i.e. a reflexive and transitive relation-

$$(0) \quad [\, x \sqsubseteq y \ \equiv\ \langle \forall z :: \ z \sqsubseteq x \Rightarrow z \sqsubseteq y \rangle \,] \quad \text{and}$$
$$(1) \quad [\, x \sqsubseteq y \ \equiv\ \langle \forall z :: \ x \sqsubseteq z \Leftarrow y \sqsubseteq z \rangle \,] \quad .$$

For a preorder $\sqsubseteq$, a value k satisfying

$$(2) \quad \langle \forall x :: [\, x \sqsubseteq k \ \equiv\ \langle \forall y : y \in W : x \sqsubseteq f.y \rangle \,] \rangle$$

is called "a highest lower bound" for the set S given by

$$S = \{ z \mid \langle \exists y : y \in W : f.y \rangle \} \quad .$$

(Note that, for f the identity function, $S = W$.)
Moreover, a value h satisfying

$$(3) \quad \langle \forall x :: [\, h \sqsubseteq x \ \equiv\ \langle \forall y : y \in W : f.y \sqsubseteq x \rangle \,] \rangle$$

is called "a lowest higher bound for S". We remind the reader that these two names only make sense because we have chosen to read the symbol $\sqsubseteq$ as "below". For the rest of this note we restrict ourselves to a preorder $\sqsubseteq$ such that highest lower and lowest higher bounds exist for any set S. Each of these bounds is unique if $\sqsubseteq$ is antisymmetric as well, i.e.

$$[x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y] \quad ;$$

a preorder that is antisymmetric as well is called "a partial order" and in what follows, $\sqsubseteq$ is further restricted to a partial order. We now show the uniqueness of the highest lower bound.

<u>Proof</u> Let k' satisfy

(2') $\quad \langle \forall x :: [x \sqsubseteq k' \equiv \langle \forall y : y \in W : x \sqsubseteq f.y \rangle] \rangle$

We then observe

$$\begin{array}{ll} & \text{true} \\ = & \{ (2) \text{ and } (2') \} \\ & \langle \forall x :: [x \sqsubseteq k \equiv x \sqsubseteq k'] \rangle \\ \Rightarrow & \{ x := k \text{ and } x := k' \} \\ & [k \sqsubseteq k \equiv k \sqsubseteq k'] \wedge [k' \sqsubseteq k \equiv k' \sqsubseteq k'] \\ = & \{ \sqsubseteq \text{ is reflexive} \} \\ & [k \sqsubseteq k'] \wedge [k' \sqsubseteq k] \\ \Rightarrow & \{ \sqsubseteq \text{ is antisymmetric} \} \\ & [k = k'] \quad . \end{array}$$

(End of Proof.)

The highest lower bound of the empty set is traditionally denoted by $\top$ and called "top"; the lowest higher bound of the empty set is traditionally denoted by $\perp$ and called "bottom". From (2) and (3) follow

$$\langle \forall x :: [x \sqsubseteq \top] \rangle \quad \text{and} \quad \langle \forall x :: [\perp \sqsubseteq x] \rangle \quad .$$

The unique value for k satisfying (2) is denoted by $\langle \sqcap y : y \in W : f.y \rangle$, i.e.

(4) $\langle \forall x :: [x \sqsubseteq \langle \sqcap y : y \in W : f.y \rangle \;\equiv\; \langle \forall y : y \in W : x \sqsubseteq f.y \rangle] \rangle$.

Similarly, the lowest higher bound is denoted using $\sqcup$:

(5) $\langle \forall x :: [\langle \sqcup y : y \in W : f.y \rangle \sqsubseteq x \;\equiv\; \langle \forall y : y \in W : f.y \sqsubseteq x \rangle] \rangle$.

* * *

Function f is monotonic means here

(6) $[x \sqsubseteq y] \Rightarrow [f.x \sqsubseteq f.y]$ for all x, y .

For monotonic f we can prove (7) and (8):

(7) $[f.\langle \sqcap y : y \in W : y \rangle \sqsubseteq \langle \sqcap y : y \in W : f.y \rangle]$

(8) $[\langle \sqcup y : y \in W : f.y \rangle \sqsubseteq f.\langle \sqcup y : y \in W : y \rangle]$,

of which we prove (7) .

<u>Proof</u> In the following we leave the ranges $x \in W, y \in W$ understood and observe for any monotonic f

$$
\begin{array}{ll}
 & [f.\langle \sqcap x :: x \rangle \sqsubseteq \langle \sqcap y :: f.y \rangle] \\
= & \quad \{ (4) \text{ with } x := f.\langle \sqcap x :: x \rangle \} \\
 & [\langle \forall y :: f.\langle \sqcap x :: x \rangle \sqsubseteq f.y \rangle] \\
= & \quad \{ \text{interchange} \} \\
 & \langle \forall y :: [f.\langle \sqcap x :: x \rangle \sqsubseteq f.y] \rangle \\
\Leftarrow & \quad \{ f \text{ is monotonic; predicate calculus} \} \\
 & \langle \forall y :: [\langle \sqcap x :: x \rangle \sqsubseteq y] \rangle \\
= & \quad \{ (4) \text{ with } x, f := \langle \sqcap x :: x \rangle, \text{identity} \} \\
 & [\langle \sqcap x :: x \rangle \sqsubseteq \langle \sqcap y :: y \rangle]
\end{array}
$$

$=\qquad \{\sqsubseteq \text{ is reflexive}\}$
$\quad true.$

(End of Proof.

* * *

We are now ready for the theorem of Knaster-Tarski: for monotonic f, the equations

(9) $\qquad x: [f.x \sqsubseteq x]$ $\qquad$ and

(10) $\qquad x: [f.x = x]$

have each a lowest solution, and the two lowest solutions are the same.

<u>Proof</u> We define q by

$$[q = \langle \sqcap y : [f.y \sqsubseteq y] : y\rangle]$$

and observe, in order to show that q solves (9)

$$
\begin{array}{ll}
 & f.q \\
= & \{\text{definition of } q\} \\
 & f.\langle \sqcap y : [f.y \sqsubseteq y] : y\rangle \\
\sqsubseteq & \{f \text{ is monotonic, (7)}\} \\
 & \langle \sqcap y : [f.y \sqsubseteq y] : f.y\rangle \\
\sqsubseteq & \{\sqcap \text{ is monotonic, see (12) below}\} \\
 & \langle \sqcap y : [f.y \sqsubseteq y] : y\rangle \\
= & \{\text{definition of } q\} \\
 & q \quad ,
\end{array}
$$

which observation establishes $[f.q \sqsubseteq q]$ thanks to the transitivity of $\sqsubseteq$; so q solves (9), and it is its lowest solution by construction, for we observe

$$\begin{array}{ll}
(11) & \langle \forall y : [f.y \sqsubseteq y] : q \sqsubseteq y \rangle \\
= & \quad \{(4), \text{ def. of } \sqcap\} \\
 & q \sqsubseteq \langle \sqcap y : [f.y \sqsubseteq y] : y \rangle \\
= & \quad \{\text{def. of } q \text{ , } \sqsubseteq \text{ is reflexive}\} \\
 & \text{true.}
\end{array}$$

Next we observe

$$\begin{array}{ll}
 & [f.q = q] \\
\Leftarrow & \quad \{\sqsubseteq \text{ is antisymmetric}\} \\
 & [f.q \sqsubseteq q] \ \wedge \ [q \sqsubseteq f.q] \\
\Leftarrow & \quad \{(11) \text{ with } y := f.q\} \\
 & [f.q \sqsubseteq q] \ \wedge \ [f.(f.q) \sqsubseteq f.q] \\
\Leftarrow & \quad \{f \text{ is monotonic}\} \\
 & [f.q \sqsubseteq q] \ \wedge \ [f.q \sqsubseteq q] \\
= & \quad \{q \text{ solves } (9)\} \\
 & \text{true}
\end{array}$$

so q solves (10) as well. Moreover, q is (10)'s lowest solution, as follows from the observation

$$\begin{array}{ll}
 & \langle \forall y : [f.y = y] : q \sqsubseteq y \rangle \\
\Leftarrow & \quad \{[f.y = y] \Rightarrow [f.y \sqsubseteq y] \text{ since } \sqsubseteq \text{ is reflexive}\} \\
 & \langle \forall y : [f.y \sqsubseteq y] : q \sqsubseteq y \rangle \\
= & \quad \{(11)\} \\
 & \text{true.}
\end{array}$$

* * *

The monotonicity of $\sqcap$ is expressed by

$$(12) \quad [\langle \forall y :: g.y \sqsubseteq h.y \rangle \Rightarrow \langle \sqcap y :: g.y \rangle \sqsubseteq \langle \sqcap y :: h.y \rangle]$$

where we left the range of y understood.

<u>Proof</u> We observe for any g, h, and range of y

$$
\begin{array}{ll}
 & \langle \sqcap y :: g.y \rangle \sqsubseteq \langle \sqcap y :: h.y \rangle \\
= & \quad \{ (0) \} \\
 & \langle \forall z :: z \sqsubseteq \langle \sqcap y :: g.y \rangle \Rightarrow z \sqsubseteq \langle \sqcap y :: h.y \rangle \rangle \\
= & \quad \{ (4) \text{ twice} \} \\
 & \langle \forall z :: \langle \forall y :: z \sqsubseteq g.y \rangle \Rightarrow \langle \forall y :: z \sqsubseteq h.y \rangle \rangle \\
\Leftarrow & \quad \{ \text{monotonicity of } \forall \} \\
 & \langle \forall z :: \langle \forall y :: z \sqsubseteq g.y \Rightarrow z \sqsubseteq h.y \rangle \rangle \\
= & \quad \{ \text{interchange} \} \\
 & \langle \forall y :: \langle \forall z :: z \sqsubseteq g.y \Rightarrow z \sqsubseteq h.y \rangle \rangle \\
= & \quad \{ (0) \} \\
 & \langle \forall y :: g.y \sqsubseteq h.y \rangle
\end{array}
$$

(End of Proof)

Notice how, in the above proof, we started using (0) and not (1), which would have introduced $\sqcap$ to the left of $\sqsubseteq$, whereas (4) enables us to eliminate $\sqcap$ to the right of $\sqsubseteq$.

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