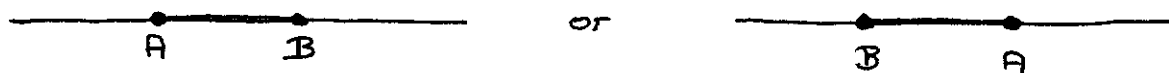


The complete $(n+1)$ -graph in n -dimensional space

We can embed the complete 2-graph with labelled vertices in a 1-dimensional world



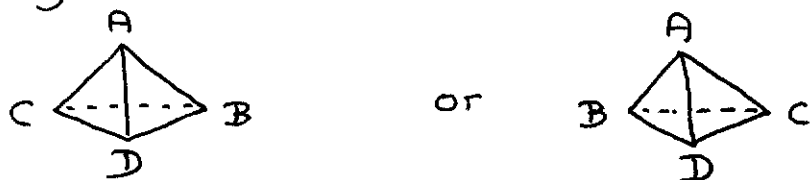
and if the two "ends" of that 1-dimensional world are distinct, the above 2 embeddings are different.

We can embed the complete 3-graph with labelled vertices in a 2-dimensional world



and if the two "sides" of the plane are distinct, so are the 2 embeddings.

Similarly - and this will be our last example - the completed 4-graph can be embedded in 2 ways in directed 3-space - i.e. a 3-dimensional world that is distinct from its mirror image -



The first question to be raised and answered is: how do we distinguish, for each number

of dimensions, between the 2 embeddings?
 Moreover we would like to do so in a manner that does not destroy the symmetry between the vertices.

Here the theory of inversions - i.e. (the number of) pairs of elements out of order - gives the answer. For any k , $k \geq 2$, the $k!$ permutations of k elements can be partitioned into 2 classes of equal size such that a single swap transforms any permutation of the one class into a permutation of the other class. For elements A, B , the classes are $\{AB\}$ and $\{BA\}$, for elements A, B, C , the classes are $\{ABC, BCA, CAB\}$ and $\{ACB, BAC, CBA\}$, for elements A, B, C, D , the one class is $\{ABCD, ACDB, ADBC, BADC, BCAD, BDCA, CABD, CBDA, CDAB, DACB, DBAC, DCBA\}$ and the construction of the other class is left to the reader. I would like to stress that the existence and "shape" of these classes have nothing to do with the labels A, B, C, \dots or their alphabetic order: they are intrinsically generated by the process of permuting. The two classes provide the answer to our first question: for any k , we associate the one class with the one embedding, and the other class with the other embedding.

The "oriented" $(n+1)$ -graph has $n+1$ "faces", one opposite to each vertex. The way to give those faces an orientation in a systematic manner is, for instance, as follows: choose from the representative class for the $(n+1)$ -graph a permutation that starts with the opposite vertex and select the rest of that permutation. So the oriented 3-graph with class $\{ABC, BCA, CAB\}$ has face BC opposite to A , face CA opposite to B and face AB opposite to C ; the 4-graph corresponding to $ABCD$ has the 4 oriented faces corresponding to BCD , ADC , ABD , and ACB respectively, and any 2 of them contain the subface they share in opposite orientation: e.g. BCD and ADC contain CD and DC respectively, the one having been obtained by truncating a leading AB (from $ABCD$, as a matter of fact), the other by truncating a leading BA (from $BADC$). Generalization to more dimensions is left to the reader.

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